

DERIVATIONS OF SIEGEL MODULAR FORMS FROM CONNECTIONS

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ABSTRACT. We introduce a method in differential geometry to study the derivative operators of Siegel modular forms. By determining the coefficients of the invariant Levi-Civita connection on a Siegel upper half plane, and further by calculating the expressions of the differential forms under this connection, we get a non-holomorphic derivative operator of the Siegel modular forms. In order to get a holomorphic derivative operator, we introduce a weaker notion, called modular connection, on the Siegel upper half plane than a connection in differential geometry. Then we show that on a Siegel upper half plane there exists at most one holomorphic modular connection in some sense, and get a possible holomorphic derivative operator of Siegel modular forms.

INTRODUCTION

In this paper, we introduce a differential geometric method to study the derivative operators of Siegel modular forms, which, theoretically, may be applied to the study of the derivative operators of any automorphic form. Our idea comes from the observation on the two derivative operators of the classical modular forms constructed by combinations. It is well-known [13] that if f is a modular forms of weight $2k$, then $D_k f := \frac{df}{dz} - \frac{\sqrt{-1}k}{y}f$ is a non-holomorphic modular forms of weight $2k+2$, and $D_k f := \frac{df}{dz} - \sqrt{-1}kG_2(z)f$, due to J.-P. Serre [9], is a holomorphic modular forms of weight $2k+2$, where $G_2(z)$ is the Eisenstein series of weight 2. We notice that the first operator can be constructed by the Levi-Civita connection corresponding to the invariant metric in the classical upper half plane, but the second can not be constructed from any connection. However, if we loosen some condition in the definition of the connection and define a concept called modular connection, we can get Serre's holomorphic derivative from the unique holomorphic modular connection on the upper half plane. In this paper we extend these results to Siegel upper half planes and Siegel modular forms. We determine the coefficients of the Levi-Civita connection corresponding to the invariant metric in a Siegel upper half plane, and compute the expressions of the differential forms under the connection, which give us a non-holomorphic derivative operator of Siegel modular forms. Our main results are as follows.

Let \mathbb{H}_g be the Siegel upper plane of degree g , $\{dZ_{ij} : 1 \leq i, j \leq g\}$ a series of coordinates on \mathbb{H}_g , $\Gamma_g = \text{Sp}(2g, \mathbb{Z})$ the full Siegel modular group which acts on \mathbb{H}_g naturally, $M_k = M_k(\Gamma_g)$ the vector space of the classical (or scalar-valued) Siegel modular forms of weight k , $\widetilde{M}_k = \widetilde{M}_k(\Gamma_g)$ the \mathbb{C}^∞ -Siegel modular forms of weight k . Put

$$\frac{\partial}{\partial Z} = (\partial_{ij})_{g \times g} \quad \text{and} \quad \partial_{ij} = \frac{1}{2^{1-\delta(i,j)}} \cdot \frac{\partial}{\partial Z_{ij}},$$

where $\delta(i, j) = 1$ if $i = j$ and $\delta(i, j) = 0$ if $i \neq j$.

Theorem 0.1 (See theorem 2.7). *Let $f \in M_{2k}(\Gamma_g)$. Then*

$$\det \left(\left[\frac{\partial}{\partial Z} - \sqrt{-1}kY^{-1} \right] f \right) \in \widetilde{M}_{2gk+2}(\Gamma_g).$$

Here Γ_g can be replaced by any congruence subgroup and the weight $2k$ can also be replaced by any positive integer, with a little modification of our proofs.

Key words and phrases. Levi-Civita connection, Siegel modular form, differential operator.

To get a holomorphic derivative operator, we introduce the notion of modular connections on the Siegel upper half plane, whose condition is weaker than the classical definition of the connections. Then we show the following result.

Theorem 0.2 (See theorem 2.9). *Any symmetric $g \times g$ matrix $G(Z) = (G_{ij}(Z))$ consisting of \mathbb{C}^∞ functions on \mathbb{H}_g , which satisfies the transformation formula*

$$(CZ + D)^{-1}G(\gamma(Z)) = G(Z) \cdot (CZ + D)^t + 2C^t$$

for any $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g, \mathbb{Z})$, gives a unique modular connection \mathbb{D} such that for any \mathbb{C}^∞ -function f on \mathbb{H}_g

$$\mathbb{D}(dZ_{rs}) = - \sum_{i,j=1}^g G_{ij} dZ_{si} dZ_{rj} \quad \text{and} \quad \mathbb{D}(f(\det(dZ))^k) = \mathrm{Tr} \left(\left[\frac{\partial}{\partial Z} - kG \right] f dZ \right) (\det(dZ))^k,$$

and thus gives a derivative operator $M_{2k} \rightarrow \widetilde{M}_{2kg+2}$ by $f \mapsto \det \left(\left[\frac{\partial}{\partial Z} - kG \right] f \right)$. Furthermore, there exists at most one holomorphic symmetric matrix G to satisfy the transformation formula. If such a G exists, the operator corresponding to G is holomorphic.

In the classical case of $g = 1$, the function $\sqrt{-1}G_2(z)$ is the unique holomorphic function on the upper half plane satisfying the condition, which gives Serre's derivative. But when $g \geq 2$ we are not able to construct such a matrix function G .

H. Maass has constructed a non-holomorphic derivative operator of Siegel modular forms by invariant differential operators. For Siegel modular forms f of weight k , Maass ([7], P317) defines the operator

$$D_k f(Z) = \det(Y)^{\kappa-k-1} \det \left(\frac{\partial}{\partial Z} \right) [\det(Y)^{k+1-\kappa} f(Z)],$$

where $\kappa = (g+1)/2$ and the determinant of $\frac{\partial}{\partial Z}$ is taken first, and shows that the differential operator D_k acts on the \mathbb{C}^∞ -Siegel modular forms and maps \widetilde{M}_k to \widetilde{M}_{k+2} . We do not know the relation between our operator in Theorem 0.1 and Maass'. Compared to our operator, D_k is linear with respect to f . Moreover, our operator is a combination of degree 1 partial derivatives of f , but D_k is a combination of degree g partial derivatives. G. Shimura [8] considers the compositions $D_r^k = D_{r+2k-2} \cdots D_{r+2} D_r$ of Maass' operator, which maps \widetilde{M}_r to \widetilde{M}_{r+2k} . For our operator one can also consider the compositions and then construct the Rankin-Cohen brackets. We wish that Maass' operator could be got in this way.

The paper is organized as follows. In section one, we introduce the concept of modular connection on a Siegel upper plane, and show several lemmas on it. In section two, we compute the expressions of the differential forms under the modular connection, and prove the two theorems above. Finally in section three, we show Lemma 2.1 which explicitly gives the connection coefficients of the Levi-Civita connection on a Siegel upper half plane.

Our calculations in sections 2 and 3 are tested by matlab in the cases $g = 2$ and $g = 3$.

1. MODULAR CONNECTIONS

In this section we first recall the definition of connections in differential geometry. Then we introduce the notion of modular connection on a Siegel upper half plane, and show several lemmas about it.

1.1. Connections in differential geometry. For the backgrounds and notations on differential geometry, especially on connections, we refer to the books [2] and [4]. Here we just recall some basic definitions and results on connections. Suppose E is a q -dimensional real vector bundle on

a smooth manifold M , and $\Gamma(E)$ is the set of smooth sections of E on M . Let $T^*(M)$ be the cotangent space of M . A connection on the vector bundle E is a map

$$D : \Gamma(E) \longrightarrow \Gamma(T^*(M) \otimes E),$$

which satisfies the following conditions

- (1) For any $s_1, s_2 \in \Gamma(E)$,

$$D(s_1 + s_2) = D(s_1) + D(s_2).$$

- (2) For any $s \in \Gamma(E)$ and any $\alpha \in \mathbb{C}^\infty(M)$,

$$D(\alpha s) = d\alpha \otimes \alpha D(s).$$

If M has a generalized Riemannian metric $G = \sum_{i,j} g_{ij} du^i du^j$, by the fundamental theorem of Riemannian geometry, M has a unique torsion-free and metric-compatible connection, called Levi-Civita connection of M . The coefficients Γ_{ij}^k of the Levi-Civita connection are given by

$$\Gamma_{ij}^k = \frac{1}{2} \sum_l g^{kl} \left(\frac{\partial g_{il}}{\partial u^j} + \frac{\partial g_{jl}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l} \right), \quad (1.1)$$

where g^{ij} are elements of the matrix $(g^{ij}) := (g_{ij})^{-1}$.

The following lemma is useful in the application of connections to automorphic forms.

Lemma 1.1. Let Γ be a group, (M, G) a Riemannian manifold and D the Levi-Civita connection on M . If Γ has a smooth left action on M such that $G(\sigma_* X, \sigma_* Y) = G(X, Y)$ for all $\sigma \in \Gamma, X, Y \in T(M)$, then

$$\sigma D = D\sigma \quad (\sigma \in \Gamma).$$

Moreover, if M is a complex manifold such that Γ maps (r, s) forms to (r, s) forms, and put $D = D^{1,0} + D^{0,1}$, where $D^{1,0}$ is the holomorphic part, then for $\sigma \in \Gamma$

$$\sigma D^{1,0} = D^{1,0} \sigma \quad \text{and} \quad \sigma D^{0,1} = D^{0,1} \sigma.$$

Proof. D is the unique torsion free connection which preserves the Riemannian metric G . Since G is Γ -invariant, the connection $\sigma^{-1} D \sigma$ also preserves the Riemannian metric and is torsion free for any $\sigma \in \Gamma$, hence $\sigma D = D\sigma$. For more detail, see ([10], P35). \square

1.2. Siegel upper half plane. We first fix some notations. The Siegel upper half plane of degree $g \geq 1$ is defined to be the $g(g+1)/2$ dimensional open complex variety

$$\mathbb{H}_g := \{Z = X + \sqrt{-1}Y \in M(g, \mathbb{C}) \mid Z^t = Z, Y > 0\}.$$

Write $Z = (Z_{ij})$. Set $\Omega = \{(i, j) \mid 1 \leq i \leq j \leq g\}$ with the dictionary order. If $I = (i, j) \in \Omega$, we define $Z_I := Z_{ij}$. Fix a series of coordinates $\{dZ_I, d\bar{Z}_I \mid I \in \Omega\}$ on \mathbb{H}_g . The symplectic group of degree $g > 0$ over \mathbb{R} is the group

$$\mathrm{Sp}(2g, \mathbb{R}) = \{M \in GL(2g, \mathbb{R}) \mid MJM^t = J\},$$

where $J = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$. We usually write an element of $\mathrm{Sp}(2g, \mathbb{R})$ in the form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A, B, C and D are $g \times g$ blocks. The symplectic group $\mathrm{Sp}(2g, \mathbb{R})$ acts on \mathbb{H}_g by the rule:

$$\gamma(Z) := (AZ + B)(CZ + D)^{-1}, \quad Z \in \mathbb{H}_g, \quad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g, \mathbb{R}).$$

By Maass ([6], P98), $d(\gamma Z) = (ZC^t + D^t)^{-1} dZ (CZ + D)^{-1} := (d\tilde{Z}_{ij})$. Let

$$(d\tilde{Z}_{11}, d\tilde{Z}_{12}, \dots, d\tilde{Z}_{1g}, \dots, \dots, d\tilde{Z}_{gg}) = (dZ_{11}, dZ_{12}, \dots, dZ_{1g}, \dots, \dots, dZ_{gg}) \cdot S(\gamma, Z)$$

where $S := S(\gamma, Z)$ is a $\frac{g(g+1)}{2} \times \frac{g(g+1)}{2}$ matrix of holomorphic functions on $\mathrm{Sp}(2g, \mathbb{Z}) \times \mathbb{H}_g$.

From Lemma 1.1, one can see that the connection matrix ω consisting of the connection coefficients of the Levi-Civita connection associated to the invariant metric $ds^2 = \text{Tr}(Y^{-1}dZ \cdot Y^{-1}d\bar{Z})$ given by Siegel ([6], P8) on the Siegel upper plane \mathbb{H}_g satisfies

$$\gamma(\omega) = -S^{-1} \cdot dS + S^{-1} \cdot \omega \cdot S$$

for all $\gamma \in \text{Sp}(2g, \mathbb{R})$. Refer also to the proof of Lemma 1.5 below. But in the studying of modular forms, we only need that the equality holds for all $\gamma \in \text{Sp}(2g, \mathbb{Z})$, the Siegel modular group. So we need to introduce a weaker notion to study modular forms.

Now we recall the definition of Siegel modular forms, for more details, see [1] and [3].

Definition 1.2. A (classical) Siegel modular form of weight k (and degree g) is a holomorphic function $f : \mathbb{H}_g \rightarrow \mathbb{C}$ such that

$$f(\gamma(Z)) = \det(CZ + D)^k f(Z)$$

for all $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2g, \mathbb{Z})$ (with the usual holomorphicity requirement at ∞ when $g = 1$).

1.3. Modular connections. The notations are the same as those above.

Definition 1.3 (Modular Connection Coefficients (MCC)). The modular connection coefficient on \mathbb{H}_g is a series of \mathbb{C}^∞ -functions $\{\Gamma_{IJ}^K \mid I, J, K \in \Omega\}$ such that for all $\gamma \in \text{Sp}(2g, \mathbb{Z})$,

$$\gamma(\omega) = -S^{-1} \cdot dS + S^{-1} \cdot \omega \cdot S, \quad \text{or} \quad S \cdot \gamma(\omega) = \omega \cdot S - dS,$$

where $\omega = (\omega_I^J)$ and $\omega_I^J = \sum_{K \in \Omega} \Gamma_{IK}^J dZ_K$. Here I and J are the row and column indices respectively. When $\{\Gamma_{IJ}^K\}$ are holomorphic, we call it holomorphic MCC (HMCC). The matrix ω is called the modular connection matrix.

In the following, $\mathbb{C}^\infty(\mathbb{H}_g)$ is the set of \mathbb{C}^∞ functions on \mathbb{H}_g , and $\text{Hol}(\mathbb{H}_g)$ is the set of holomorphic functions on \mathbb{H}_g .

Definition 1.4 (Modular Connection). Let $\{\Gamma_{IJ}^K\}$ be a MCC (resp. HMCC) on \mathbb{H}_g and Ω^∞ be the commutative $\mathbb{C}^\infty(\mathbb{H}_g)$ -algebra (resp. $\text{Hol}(\mathbb{H}_g)$ -algebra) generated by $\{dZ_I\}_{I \in \Omega}$ with the relations $dZ_I dZ_J = dZ_J dZ_I$ for any $I, J \in \Omega$. The linear operator

$$D : \Omega^\infty \longrightarrow \Omega^\infty$$

is uniquely defined by the following two relations

$$D(dZ_K) = - \sum_{I, J \in \Omega} \Gamma_{IJ}^K dZ_I dZ_J$$

and

$$D(f dZ_{K_1} dZ_{K_2} \cdots dZ_{K_r}) = df \cdot dZ_{K_1} \cdots dZ_{K_r} + \sum_{i=1}^r f dZ_{K_1} dZ_{K_2} \cdots D(dZ_{K_i}) \cdots dZ_{K_r},$$

and we call it the modular connection associated to $\{\Gamma_{IJ}^K\}$.

One can easily show that $D(dZ_K) = - \sum_{I \in \Omega} \omega_I^K \cdot dZ_I$ and

$$(D(dZ_{11}), D(dZ_{12}), \cdots, \cdots, \cdots, D(dZ_{gg})) = -(dZ_{11}, dZ_{12}, \cdots, \cdots, \cdots, dZ_{gg}) \cdot \omega$$

When $\{\Gamma_{IJ}^K\}$ is holomorphic, we also call D a holomorphic modular connection. Compared with the definition of connections in differential geometry, except for the weaker conditions, the modular connection also ignore the part on $\{d\bar{Z}_I\}_{I \in \Omega}$.

1.4. Basic lemmas on modular connections. The following two lemmas are basic to our application of modular connections to the Siegel modular forms. For the modular connections, we have the similar result to Lemma 1.1.

Lemma 1.5. Let D be a modular connection on \mathbb{H}_g . Then $\gamma D = D\gamma$ for any $\gamma \in \mathrm{Sp}(2g, \mathbb{Z})$. Moreover, if f is a Siegel modular forms of weight $2k$, then $D(f(\det(dZ))^k)$ is invariant under the action of $\Gamma_g = \mathrm{Sp}(2g, \mathbb{Z})$.

Proof. Let $\alpha = (dZ_{11}, \dots, dZ_{1g}, dZ_{22}, \dots, dZ_{2g}, \dots, dZ_{gg})$. Then $D\alpha = -\alpha\omega$. On one side,

$$\gamma(D\alpha) = -\gamma(\alpha)\gamma(\omega) = -\alpha S(-S^{-1}dS + S^{-1} \cdot \omega \cdot S) = \alpha(dS - \omega \cdot S).$$

On the other side,

$$D(\gamma\alpha) = D(\alpha \cdot S) = D(\alpha) \cdot S + \alpha \cdot dS = -\alpha\omega \cdot S + \alpha dS = \alpha(-\omega \cdot S + dS).$$

For $f \in M_{2k}(\Gamma_g)$, we see $f(\det(dZ))^k$ is invariant under the action of Γ_g , and so is $D(f(\det(dZ))^k)$. \square

The following lemma directly from Lemma 1.1 gives a modular connection.

Lemma 1.6. The holomorphic part $D^{1,0}$ of the Levi-Civita connection associated to the invariant metric $ds^2 = \mathrm{Tr}(Y^{-1}dZ \cdot Y^{-1}d\bar{Z})$ on the Siegel upper plane \mathbb{H}_g is a modular connection.

We will give the explicit expression of $D^{1,0}(f \det(dZ)^k)$ in Proposition 2.4.

1.5. Modular connections in the classical case. Let us consider the classical upper half plane $\mathbb{H} = \mathbb{H}_1$ to look for what condition of the coefficient $\Gamma := \Gamma_{(1,1),(1,1)}^{(1,1)}$ of a modular connection should satisfy. Let $\omega = \Gamma dz$. One can easily check that for $\gamma \in \mathrm{SL}(2, \mathbb{Z})$

$$\gamma(\omega) = -S^{-1}dS + S^{-1}\omega S \iff \frac{\gamma(\Gamma)}{(cz+d)^2} = \Gamma + \frac{2c}{cz+d}.$$

Recall that ([5], P113)

$$\frac{1}{(cz+d)^2} \cdot \frac{\sqrt{-1}}{\mathrm{Im}(\gamma z)} = \frac{\sqrt{-1}}{\mathrm{Im}(z)} + \frac{2c}{cz+d} \quad \text{and} \quad \frac{\sqrt{-1}G_2(\gamma z)}{(cz+d)^2} = \sqrt{-1}G_2(z) + \frac{2c}{cz+d},$$

where

$$G_2(z) = \frac{1}{2\pi} \left(\sum_{n \neq 0} \frac{1}{n^2} + \sum_{m \neq 0} \sum_{n \in \mathbb{Z}} \frac{1}{(mz+n)^2} \right).$$

So $\frac{\sqrt{-1}}{y}$ and $\sqrt{-1}G_2(z)$ give us two modular connections on \mathbb{H} , which we denote by D_1 and D_2 respectively. The later is holomorphic. We have

$$D_1(fdz) = \left(\frac{df}{dz} - \frac{\sqrt{-1}}{y}f \right) dzdz \quad \text{and} \quad D_2(fdz) = \left(\frac{df}{dz} - \sqrt{-1}G_2(z)f \right) dzdz.$$

By the expressions of $D_1(f(dz)^k)$ and $D_2(f(dz)^k)$ and by Lemma 1.5, we have

Corollary 1.7. Let f be a modular form of weight $2k$. Then $\frac{df}{dz} - \frac{\sqrt{-1}k}{y}f$ and $\frac{df}{dz} - \sqrt{-1}kG_2(z)f$ are modular forms of weight $2k+2$. They are non-holomorphic and holomorphic, respectively.

In fact, the modular connection D_1 comes from the Levi-Civita connection D associated to the invariant metric $ds^2 = \frac{dz d\bar{z}}{y^2}$.

Lemma 1.8. $D(dz) = -\frac{\sqrt{-1}}{y}dz dz$ and $D(d\bar{z}) = \frac{\sqrt{-1}}{y}d\bar{z} d\bar{z}$. So the coefficients of D give the modular connection D_1 .

Proof. Since $ds^2 = \frac{dx^2 + dy^2}{y^2} = \frac{dz d\bar{z}}{y^2}$, using the coordinates $dz, d\bar{z}$ and the equality (1.1), we have $\Gamma_{1,1}^1 = \frac{\sqrt{-1}}{y}$, $\Gamma_{2,1}^1 = \Gamma_{1,2}^1 = 0$, $\Gamma_{2,1}^2 = \Gamma_{1,2}^2 = 0$ and $\Gamma_{2,2}^2 = -\frac{\sqrt{-1}}{y}$. \square

The connection D_2 is not from differential geometry. It is unique.

Lemma 1.9 (Uniqueness Lemma). $\sqrt{-1}G_2(z)$ is the unique holomorphic function Γ satisfying

$$\frac{\gamma(\Gamma)}{(cz+d)^2} = \Gamma + \frac{2c}{cz+d} \text{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}),$$

and so D_2 is the unique holomorphic modular connection on \mathbb{H} .

Proof. We have $\gamma(\sqrt{-1}G_2(z) - \Gamma) = (cz+d)^2(\sqrt{-1}G_2(z) - \Gamma)$ for all $\gamma \in \text{SL}(2, \mathbb{Z})$. So $\sqrt{-1}G_2(z) - \Gamma$ is a modular form of weight 2 and hence must be zero ([5], P117). \square

We will generalize these results to \mathbb{H}_g .

2. DERIVATIVE OPERATORS OF SIEGEL MODULAR FORMS

In this section, we first state the result determining the coefficients of the invariant Levi-Civita connection on a Siegel upper half plane, whose proof we put in the last section, then we compute the expressions of the differential forms under this connection. Finally we get a non-holomorphic derivative operator and a possible holomorphic derivative operator.

2.1. Coefficients of Levi-Civita connection. The notations are the same as those in section 1. For $I = (i, j) \in \Omega$, put

$$N(I) = \frac{(i-1)(2g-i)}{2} + j,$$

which gives a one to one and order keeping correspondence between Ω and $\{1, 2, \dots, \frac{g(g+1)}{2}\}$. Write $u^{N(I)} = Z_{ij}$ and $u^{N(g,g)+N(i,j)} = \bar{Z}_{ij}$. For $Z = X + \sqrt{-1}Y \in \mathbb{H}_g$, let $R = (R_{ij}) = Y^{-1}$. For $1 \leq s \leq g$, we set

$$\Omega_s = \{(1, s), (2, s), \dots, (s, s), (s, s+1), \dots, (s, g)\} \subset \Omega.$$

Let $K = (r, s) \in \Omega$. Assume that the elements of Z in the column including $Z_K = Z_{rs}$ are

$$u^{a_1} = Z_{1s}, u^{a_2} = Z_{2s}, \dots, u^{a_s} = Z_{ss}, u^{a_{s+1}} = Z_{s+1,s} = Z_{s,s+1}, \dots, u^{a_g} = Z_{gs} = Z_{sg},$$

and the elements in the row including Z_{rs} are

$$u^{b_1} = Z_{r1} = Z_{1r}, u^{b_2} = Z_{r2} = Z_{2r}, \dots, u^{b_r} = Z_{rr}, u^{b_{r+1}} = Z_{r,r+1}, \dots, u^{b_g} = Z_{rg}.$$

For $I \times J \in \Omega_s \times \Omega_r$, assume $Z_I = u^{a_i}$ and $Z_J = u^{b_j}$. Similarly do it for $J \times I \in \Omega_s \times \Omega_r$. We define

$$\Gamma_{IJ}^K (= \Gamma_{JI}^K) := \begin{cases} \frac{\sqrt{-1}R_{ij}}{2^{(1-\delta(r,s))(1-\delta(a_i, b_j))}} & \text{if } I \times J \text{ or } J \times I \in \Omega_s \times \Omega_r \\ 0 & \text{if } I \times J \text{ and } J \times I \notin \Omega_s \times \Omega_r, \end{cases} \quad (2.1)$$

where $\delta(r, s)$ denotes the Kronecker delta symbol. For example, $\Gamma_{(1,i)(1,j)}^{(1,1)} = \sqrt{-1}R_{ij}$ and $\Gamma_{IJ}^{(1,1)} = 0$ if I or $J \notin \Omega_1$.

Notice that if we use the coordinates $\{u^{a_i}, u^{b_j}\}$, then the coefficients of the Levi-Civita connection satisfy

$$Z\Gamma_{IJ}^K = u\Gamma_{N(I)N(J)}^{N(K)}.$$

In this paper we will use these two kinds of coordinates alternately. The proof of the following lemma is long and complicated. For the convenience of the reader, we put it in the last section.

Lemma 2.1. The coefficients $\{\Gamma_{IJ}^K\}$ defined in the equality (2.1) give the Levi-Civita connection on \mathbb{H}_g associated to the invariant metric $ds^2 = \text{Tr}(Y^{-1}dZ Y^{-1}d\bar{Z})$, and hence give a modular connection, which we denote by D .

2.2. Expression of differential forms under D . We first compute $D(dZ_K)$.

Lemma 2.2. Let $K = (r, s) \in \Omega$. We have

$$D(dZ_K) = -\sqrt{-1}(dZ_{s1}, dZ_{s2}, \dots, dZ_{sg})Y^{-1} \cdot (dZ_{r1}, dZ_{r2}, \dots, dZ_{rg})^t.$$

Proof. The notations are as above. If $r = s$, then $\Omega_r = \Omega_s$. By Lemma 2.1, we have

$$\Gamma_{IJ}^K = \Gamma_{JI}^K = \begin{cases} \sqrt{-1}R_{ij} & \text{if } I \text{ and } J \in \Omega_r \\ 0 & \text{if } I \text{ or } J \notin \Omega_r, \end{cases}$$

Hence,

$$\begin{aligned} D(dZ_K) &= - \sum_{I, J \in \Omega} \Gamma_{IJ}^K dZ_I dZ_J = - \sum_{I, J \in \Omega_r} \Gamma_{IJ}^K dZ_I dZ_J = - \sum_{i, j=1}^g \sqrt{-1}R_{ij} dZ_{si} dZ_{rj} \\ &= -\sqrt{-1}(dZ_{s1}, dZ_{s2}, \dots, dZ_{sg})Y^{-1} \cdot (dZ_{r1}, dZ_{r2}, \dots, dZ_{rg})^t. \end{aligned}$$

If $r \neq s$, we assume $r < s$. Then $\Omega_r \cap \Omega_s = \{(r, s)\}$ and $(\Omega_r \times \Omega_s) \cap (\Omega_s \times \Omega_r) = \{(r, s) \times (r, s)\}$. Put $A = (\Omega_r \times \Omega_s) \cup (\Omega_s \times \Omega_r)$ and $B = (\Omega_r \times \Omega_s) \cap (\Omega_s \times \Omega_r)$. We have again by Lemma 2.1,

$$\Gamma_{IJ}^K = \Gamma_{JI}^K = \begin{cases} \frac{\sqrt{-1}R_{ij}}{2^{1-\delta(a_i, b_j)}} = \frac{\sqrt{-1}R_{ij}}{2} & \text{if } I \times J \in A \setminus B \\ \frac{\sqrt{-1}R_{ij}}{2^{1-\delta(a_i, b_j)}} = \sqrt{-1}R_{ij} & \text{if } I \times J \in B. \\ 0 & \text{if } I \times J \notin A \end{cases}$$

Hence,

$$\begin{aligned} D(dZ_K) &= - \sum_{I, J \in \Omega} \Gamma_{IJ}^K dZ_I dZ_J = - \sum_{I \times J \in A} \Gamma_{IJ}^K dZ_I dZ_J \\ &= -2 \sum_{\substack{I \in \Omega_r \\ J \in \Omega_s}} \Gamma_{IJ}^K dZ_I dZ_J + \sum_{\substack{I \in \Omega_r \cap \Omega_s \\ J \in \Omega_r \cap \Omega_s}} \Gamma_{IJ}^K dZ_I dZ_J = -2 \sum_{I \times J \in \Omega_r \times \Omega_s - B} \Gamma_{IJ}^K dZ_I dZ_J - \sum_{I \times J \in B} \Gamma_{IJ}^K dZ_I dZ_J \\ &= - \sum_{i, j=1}^g \sqrt{-1}R_{ij} dZ_{si} dZ_{rj} = -\sqrt{-1}(dZ_{s1}, dZ_{s2}, \dots, dZ_{sg})Y^{-1} \cdot (dZ_{r1}, dZ_{r2}, \dots, dZ_{rg})^t. \end{aligned}$$

The case $s < r$ is similar. □

Proposition 2.3. $D(\det(dZ)) = -\sqrt{-1}\text{Tr}(Y^{-1}dZ) \det(dZ)$.

Proof. Put $\alpha_i = (dZ_{i1}, dZ_{i2}, \dots, dZ_{ig})$ and $\beta_j = (dZ_{1j}, dZ_{2j}, \dots, dZ_{gj})^t$. By Lemma 2.2, $D(d(Z_{ij})) = -\sqrt{-1}\alpha_i Y^{-1}\beta_j$ and $D(\alpha_i) = -\sqrt{-1}\alpha_i Y^{-1}dZ$. Thus

$$D(\det(dZ)) = -\sqrt{-1} \left\{ \det \begin{pmatrix} \alpha_1 Y^{-1} dZ \\ \alpha_2 \\ \vdots \\ \alpha_g \end{pmatrix} + \det \begin{pmatrix} \alpha_1 \\ \alpha_2 Y^{-1} dZ \\ \vdots \\ \alpha_g \end{pmatrix} + \dots + \det \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{g-1} \\ \alpha_g Y^{-1} dZ \end{pmatrix} \right\}.$$

Let $A = Y^{-1}dZ = (A_{ij})$ and $dZ[i, j]$ be the algebraic cofactor of dZ at the position (i, j) . By the formula above, we have

$$\begin{aligned} \sqrt{-1}D(\det(dZ)) &= \sum_{k, j=1}^n dZ_{ki} \cdot A_{ij} \cdot dZ[k, j] = \sum_{j, i=1}^n A_{ij} \sum_{k=1}^n dZ_{ik} \cdot dZ[k, j] \\ &= \sum_{j, i=1}^n A_{ij} \delta(i, j) \det(dZ) = \text{Tr}(A) \det(dZ) = \text{Tr}(Y^{-1}dZ) \det(dZ). \end{aligned}$$

□

Put

$$\frac{\partial}{\partial Z} = (\partial_{ij})_{g \times g}, \quad \partial_{ij} = \frac{1 + \delta(i, j)}{2} \cdot \frac{\partial}{\partial Z_{ij}}$$

as in the introduction. Then for any \mathbb{C}^∞ -function f on \mathbb{H}_g ,

$$df = \sum_{1 \leq i \leq j \leq g} \frac{\partial f}{\partial Z_{ij}} dZ_{ij} = \sum_{i=1}^g \sum_{j=1}^g \partial_{ij} f dZ_{ij} = \text{Tr} \left(\frac{\partial}{\partial Z} f \cdot dZ \right)$$

Proposition 2.4. For any \mathbb{C}^∞ -function f on \mathbb{H}_g , we have

$$D \left(f \det(dZ)^k \right) = \text{Tr} \left(\left[\frac{\partial}{\partial Z} - \sqrt{-1} k Y^{-1} \right] f dZ \right) \det(dZ)^k.$$

Proof. Since $df = \text{Tr} \left(\frac{\partial}{\partial Z} f \cdot dZ \right)$, we have, by Proposition 2.3,

$$\begin{aligned} D \left(f \det(dZ)^k \right) &= df \cdot \det(dZ)^k + f \cdot D((\det(dZ))^k) = (df - \sqrt{-1} k f \text{Tr}(Y^{-1} dZ)) \det(dZ)^k \\ &= \text{Tr} \left(\left[\frac{\partial}{\partial Z} - \sqrt{-1} k Y^{-1} \right] f dZ \right) \det(dZ)^k. \end{aligned}$$

□

In the following, $\Omega_{\mathbb{H}_g}^i$ is the sheaf of holomorphic i -forms on \mathbb{H}_g . Recall that a section of $\Omega_{\mathbb{H}_g}^1$ can be written as $\text{Tr}(GdZ)$, where G is a symmetric matrix of holomorphic functions on \mathbb{H}_g .

Proposition 2.5. For any section $\text{Tr}(GdZ) \in \Omega_{\mathbb{H}_g}^1$, where G is a symmetric matrix of holomorphic functions on \mathbb{H}_g , we have

$$D(\text{Tr}(GdZ)) = \text{Tr} \left\{ \left[\left(\frac{\partial}{\partial Z} \right)^t \otimes G \right] \cdot [dZ \otimes dZ] \right\} - \sqrt{-1} \text{Tr}(GdZ \cdot Y^{-1} dZ),$$

where \otimes is the Kronecker product of matrices.

To show this proposition, we need the following lemma.

Lemma 2.6. We have

(1) Let $A = (a_{ij})_{n \times n}$, $B = (b_{ij})_{n \times n}$, $C = (c_{ij})_{n \times n}$, $D = (d_{ij})_{n \times n}$. Then

$$\text{Tr}((A \otimes B)(C \otimes D)) = \sum_{i,j,k,l=1}^n a_{ij} b_{kl} c_{lk} d_{ji} = \text{Tr}((A \otimes C)(B \otimes D)),$$

(2) $d(\text{Tr}(GdZ)) = \text{Tr} \left(\left(\left(\frac{\partial}{\partial Z} \right)^t \otimes G \right) \cdot (dZ \otimes dZ) \right)$ for a symmetric matrix $G = (G_{ij})$ of functions.

Proof. The proof of (1) is easy. We only show (2). By (1),

$$\begin{aligned} d(\text{Tr}(GdZ)) &= d \left(\sum_{i,j=1}^g G_{ij} dZ_{ij} \right) = \sum_{i,j=1}^g \sum_{k,l=1}^g \partial_{kl} G_{ij} \cdot dZ_{kl} dZ_{ji} \\ &= \text{Tr} \left(((\partial_{kl})^t \otimes (G_{ij})) \cdot (dZ \otimes dZ) \right). \end{aligned}$$

□

Proof of Proposition 2.5. As $dZ = (\alpha_1^t, \alpha_2^t, \dots, \alpha_g^t)^t = (\beta_1, \beta_2, \dots, \beta_g)$, we have

$$\begin{aligned} & D(\text{Tr}(GdZ)) - \text{Tr} \left\{ \left[\left(\frac{\partial}{\partial Z} \right)^t \otimes G \right] \cdot [dZ \otimes dZ] \right\} = \text{Tr}(GD(dZ)) \\ &= \sum_{i,j=1}^g G_{ij} D(dZ_{ij}) = -\sqrt{-1} \sum_{i,j=1}^g G_{ij} d(\alpha_i) Y^{-1} d(\beta_j) = -\sqrt{-1} \text{Tr}(GdZY^{-1}dZ). \end{aligned}$$

□

2.3. Derivative operators. If $0 \neq f \in M_{2k}(\Gamma_g)$, we have, by Proposition 2.4 and Lemma 1.5,

$$\text{Tr} \left(\left[\frac{\partial}{\partial Z} - \sqrt{-1}kY^{-1} \right] f dZ \right) \det(dZ)^k$$

is invariant under the action of Γ_g . Let $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$. Since $\gamma(\det(dZ))^k = \frac{\det(dZ)^k}{\det(CZ+D)^{2k}}$ and $\gamma(f) = \det(CZ+D)^{2k} f$, we have

$$\text{Tr} \left(\frac{1}{f} \left[\frac{\partial}{\partial Z} - \sqrt{-1}kY^{-1} \right] f dZ \right)$$

is invariant under Γ_g . Put $h := \frac{1}{f} \left(\frac{\partial}{\partial Z} - \sqrt{-1}kY^{-1} \right) f$. Then h is a symmetric matrix of functions on \mathbb{H}_g , and $\text{Tr}(hdZ)$ is invariant under the action of Γ_g . Since for any $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$, $d(\gamma Z) = (ZC^t + D^t)^{-1} \cdot dZ \cdot (CZ + D)^{-1}$, we have (see ([3], P210))

$$h(\gamma Z) = (CZ + D)h(Z)(ZC^t + D^t).$$

Thus $\det(h) = \det \left(\frac{1}{f} \left(\frac{\partial}{\partial Z} - \sqrt{-1}kY^{-1} \right) f \right)$ is a non-holomorphic Siegel modular form of weight 2. Let $\widetilde{M}_k(\Gamma_g)$ be the \mathbb{C}^∞ -Siegel modular forms of weight k as in the introduction. Finally we get

Theorem 2.7. *If $f \in M_{2k}(\Gamma_g)$, then $\det \left(\left(\frac{\partial}{\partial Z} - \sqrt{-1}kY^{-1} \right) f \right) \in \widetilde{M}_{2kg+2}(\Gamma_g)$.*

Let $f \in M_{2r}(\Gamma_g)$ and $h \in M_{2s}(\Gamma_g)$. We have

$$\begin{aligned} & D(f \det(dZ)^r) \cdot h \det(dZ)^s - f \det(dZ)^r \cdot D(h \det(dZ)^s) \\ &= \text{Tr} \left(\left[h \frac{\partial}{\partial Z} f - f \frac{\partial}{\partial Z} h \right] dZ \right) \det(dZ)^{r+s} \end{aligned}$$

is invariant under Γ_g . The same consideration as above gives us

$$\det \left(h \frac{\partial}{\partial Z} f - f \frac{\partial}{\partial Z} h \right) \in M_{2(r+s)g+2}.$$

We can continue this construction to find combinations of higher derivatives of f and h which are modular. By setting $[f, h]_0 := fh$, $[f, h]_1 := \det(h \frac{\partial}{\partial Z} f - f \frac{\partial}{\partial Z} h)$, and so on, one would get the Rankin-Cohen brackets.

2.4. The unique theorem. We first show a lemma.

Lemma 2.8. Let $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2g, \mathbb{Z})$. Then

$$(CZ + D)^{-1} \sqrt{-1}(\text{Im} \gamma(Z))^{-1} = \sqrt{-1}(\text{Im}(Z))^{-1} \cdot (CZ + D)^t + 2C^t.$$

Proof. Since $\text{Im}(\gamma(Z)) = ((C\bar{Z} + D)^t)^{-1}Y(CZ + D)^{-1}$ (see [1]) and $Y^t = Y$, we have

$$\begin{aligned} (\text{Im}\gamma(Z))^{-1} &= (CZ + D)Y^{-1} \cdot (C\bar{Z} + D)^t = (CZ + D)Y^{-1} \cdot (CZ + D - 2\sqrt{-1}CY)^t \\ &= (CZ + D)Y^{-1} \cdot (CZ + D)^t - (CZ + D)2\sqrt{-1}C^t, \end{aligned}$$

and thus the result. \square

Theorem 2.9. *For any symmetric $g \times g$ matrix $G = (G_{ij})$ consisting of \mathbb{C}^∞ (or holomorphic) functions on \mathbb{H}_g which satisfies the transformation formula*

$$(CZ + D)^{-1}\gamma(G) = G \cdot (CZ + D)^t + 2C^t,$$

there exists a unique modular connection \mathbb{D} such that

$$\mathbb{D}(dZ_{rs}) = - \sum_{i,j=1}^g G_{ij} dZ_{si} dZ_{rj} \quad \text{and} \quad \mathbb{D}(f(\det(dZ))^k) = \text{Tr} \left(\left[\frac{\partial}{\partial \bar{Z}} - kG \right] f dZ \right) (\det(dZ))^k,$$

and thus G gives a derivative operator $M_{2k} \rightarrow \widetilde{M}_{2k+2}$ by $f \mapsto \det \left(\left[\frac{\partial}{\partial \bar{Z}} - kG \right] f \right)$. Furthermore, there exists at most one holomorphic symmetric matrix G to satisfy the transformation formula. If such a G exists, the operator corresponding to G is holomorphic.

Proof. One notes that in the definition of modular connection coefficients Γ_{IJ}^K , we only need the transformation law $\gamma(\omega) = -S^{-1} \cdot dS + S^{-1} \cdot \omega \cdot S$ for $\gamma \in \text{Sp}(2g, \mathbb{Z})$. If G has the same transformation law as $\sqrt{-1}(\text{Im}(Z))^{-1}$, then we can use the same method in Lemma 2.1 to construct $\{\Gamma_{IJ}^K\}$ and to calculate the expressions of the differential forms under \mathbb{D} . The same discussion as in subsection 2.3 tells us $\det \left(\left[\frac{\partial}{\partial \bar{Z}} - kG \right] f \right) \in \widetilde{M}_{2k+2}$.

On the uniqueness, let G and \tilde{G} be two holomorphic matrices to satisfy the transformation formula. Then

$$(CZ + D)^{-1}(G(\gamma Z) - \tilde{G}(\gamma Z)) = (G(Z) - \tilde{G}(Z))(CZ + D)^t.$$

So $\text{Tr}\{(G - \tilde{G})dZ\} \in (\Omega_{\mathbb{H}_g}^1)^{\Gamma_g}$. In [11] and [12] R. Weissauer proved that if v is not of the form $[u] := ug - \frac{1}{2}u(u-1)$, then $(\Omega_{\mathbb{H}_g}^v)^{\Gamma_g} = 0$. If $g \geq 2$, one gets $(\Omega_{\mathbb{H}_g}^1)^{\Gamma_g} = 0$, and thus $0 = \text{Tr}\{(G - \tilde{G})dZ\} = 2 \sum_{i < j} (G_{ij} - \tilde{G}_{ij})dZ_{ji} + \sum_{i=1}^g (G_{ii} - \tilde{G}_{ii})dZ_{ii}$. Hence $G_{ij} = \tilde{G}_{ij}$ for all $1 \leq i, j \leq g$. The case $g = 1$ has been proved in Lemma 1.9. \square

Question 2.10. Does there exist such a G ? If so, how to construct it?

3. EXPLICIT CONSTRUCTION OF THE LEVI-CIVITA CONNECTION

In this section we give the proof of Lemma 2.1. We denote I, J, K, L, \dots the elements in Ω , i, j, k, l, r, s, \dots the elements in $\{1, 2, \dots, g\}$ and $\alpha, \beta, \gamma, \delta, \epsilon$ the elements in $\{1, 2, \dots, g(g+1)/2\}$.

3.1. Riemannian metric. Let $R := (R_{ij})_{g \times g} = Y^{-1}$. Then

$$ds^2 = \text{Tr}(R \cdot dZ \cdot R \cdot d\bar{Z}) = \sum_{i \leq j} \sum_{r \leq s} 2^{2-\delta(i,j)-\delta(r,s)} \times \frac{R_{ir}R_{js} + R_{jr}R_{is}}{2} dZ_{ij} d\bar{Z}_{rs},$$

and thus the Riemannian metric matrix associated to $ds^2 = \text{Tr}(Y^{-1}dZ \cdot Y^{-1}d\bar{Z})$ is given by

$$G = \begin{pmatrix} 0 & W \\ W & 0 \end{pmatrix}$$

where

$$W = (W_{IJ})_{I, J \in \Omega}, \quad W_{IJ} = \frac{R_{ir}R_{js} + R_{jr}R_{is}}{2^{\delta(i,j)+\delta(r,s)}}, \quad \text{if } I = (i, j) \text{ and } J = (r, s).$$

Lemma 3.1. The inverse W^{-1} of W is given by

$$M := (M_{IJ})_{I,J \in \Omega}, \quad M_{IJ} = Y_{ir}Y_{js} + Y_{jr}Y_{is}, \quad \text{if } I = (i, j) \text{ and } J = (r, s).$$

Proof. We need to show that for any $I = (i, j) \in \Omega$ and $K = (p, q) \in \Omega$,

$$\sum_{J \in \Omega} M_{IJ} W_{JK} = \delta(I, K).$$

By direct computations, we have

$$\begin{aligned} & \sum_{1 \leq r \leq s \leq g} M_{(i,j),(r,s)} W_{(r,s),(p,q)} = \sum_{1 \leq r \leq s \leq g} (Y_{ir}Y_{js} + Y_{jr}Y_{is}) \frac{R_{rp}R_{sq} + R_{sp}R_{rq}}{2^{\delta(r,s)+\delta(p,q)}} \\ &= \sum_{1 \leq r \leq g} 2^{1-\delta(p,q)} Y_{ir}Y_{jr} R_{rp}R_{rq} + \sum_{1 \leq r < s \leq g} (Y_{ir}Y_{js} + Y_{jr}Y_{is}) \frac{R_{rp}R_{sq} + R_{sp}R_{rq}}{2^{\delta(p,q)}} \\ &= 2^{-\delta(p,q)} \sum_{1 \leq r \leq g} \sum_{1 \leq s \leq g} (Y_{ir}Y_{js} R_{rp}R_{sq} + Y_{jr}Y_{is} R_{rp}R_{sq}) \\ &= 2^{-\delta(p,q)} \sum_{1 \leq r \leq g} (Y_{ir} R_{rp} \delta(j, q) + Y_{jr} R_{rp} \delta(i, q)) \\ &= 2^{-\delta(p,q)} \{ \delta(i, p) \delta(j, q) + \delta(j, p) \delta(i, q) \} = \delta(I, K). \end{aligned}$$

Notice that, in the last three steps, we have used the equality $\sum_{1 \leq r \leq g} Y_{ir} R_{rp} = \delta(r, p)$, which comes from $R = Y^{-1}$. \square

3.2. Connection Coefficients. As before, we put $u^{N(i,j)} = Z_{ij}$, $u^{N(g,g)+N(i,j)} = \bar{Z}_{ij}$,

$$G = \begin{pmatrix} 0 & W \\ W & 0 \end{pmatrix} \quad \text{and} \quad \hat{G} := G^{-1} = \begin{pmatrix} 0 & M \\ M & 0 \end{pmatrix}.$$

By the Equality (1.1), we have

$$\Gamma_{\alpha,\beta}^{\gamma} = \sum_{1 \leq \rho \leq g(g+1)} \frac{1}{2} \hat{G}_{\gamma,\rho} \left(\frac{\partial G_{\alpha,\rho}}{\partial u^{\beta}} + \frac{\partial G_{\beta,\rho}}{\partial u^{\alpha}} - \frac{\partial G_{\alpha,\beta}}{\partial u^{\rho}} \right).$$

Assume $0 \leq \alpha, \beta, \gamma \leq \frac{g(g+1)}{2}$, then $G_{\alpha,\beta} = 0$. We have:

$$\Gamma_{\alpha,\beta}^{\gamma} = \sum_{1 \leq \rho \leq g(g+1)} \frac{1}{2} \hat{G}_{\gamma,\rho} \left(\frac{\partial G_{\alpha,\rho}}{\partial u^{\beta}} + \frac{\partial G_{\beta,\rho}}{\partial u^{\alpha}} \right).$$

Hence for $I, J, K \in \Omega$

$$\Gamma_{I,J}^K = \sum_{L \in \Omega} \frac{1}{2} M_{K,L} \left(\frac{\partial W_{I,L}}{\partial Z_J} + \frac{\partial W_{J,L}}{\partial Z_I} \right).$$

Again, notice that

$$\sum_{L \in \Omega} M_{K,L} W_{I,L} = \delta(K, I) \quad \text{and} \quad \sum_{L \in \Omega} M_{K,L} W_{J,L} = \delta(K, J).$$

Do partial derivatives on both sides with respect to Z_J and Z_I respectively, we have

$$\sum_{L \in \Omega} M_{K,L} \frac{\partial W_{I,L}}{\partial Z_J} + \sum_{L \in \Omega} \frac{\partial M_{K,L}}{\partial Z_J} W_{I,L} = 0,$$

and

$$\sum_{L \in \Omega} M_{K,L} \frac{\partial W_{J,L}}{\partial Z_I} + \sum_{L \in \Omega} \frac{\partial M_{K,L}}{\partial Z_I} W_{J,L} = 0.$$

Finally we get

$$\Gamma_{I,J}^K = -\frac{1}{2} \left(\sum_{L \in \Omega} \frac{\partial M_{K,L}}{\partial Z_J} W_{I,L} + \sum_{L \in \Omega} \frac{\partial M_{K,L}}{\partial Z_I} W_{J,L} \right).$$

If $I = (i, j), J = (r, s), K = (p, q)$ and $L = (a, b) \in \Omega$, then $M_{I,J} = Y_{ir}Y_{js} + Y_{jr}Y_{is}$ and

$$\begin{aligned} \frac{\partial M_{K,L}}{\partial Z_J} &= \frac{\partial(Y_{pa}Y_{qb} + Y_{qa}Y_{pb})}{\partial Z_J} \\ &= -\frac{\sqrt{-1}}{2} \{ \sigma_{(p,a),(r,s)} Y_{qb} + \sigma_{(q,b),(r,s)} Y_{pa} + \sigma_{(q,a),(r,s)} Y_{pb} + \sigma_{(p,b),(r,s)} Y_{qa} \} \end{aligned}$$

Here we define:

$$\sigma_{(p,a),(r,s)} = \begin{cases} 1, & \text{if } Z_{pa} = Z_{rs}, \\ 0, & \text{if } Z_{pa} \neq Z_{rs}. \end{cases}$$

One should notice the difference of the notation above with the notation $\delta_{(p,a),(r,s)} := \delta((p, a), (r, s)) = \delta(p, r)\delta(a, s)$. These two notations have the following relations:

$$\sigma_{(p,a),(r,s)} = \delta_{(p,a),(r,s)} + \delta_{(p,a),(s,r)} - \delta_{(p,a),(r,s)} \cdot \delta_{(p,a),(s,r)}.$$

Using the equality $W_{IJ} = \frac{R_{ir}R_{js} + R_{jr}R_{is}}{2^{\delta(i,j) + \delta(r,s)}}$ and others above, we have

$$\begin{aligned} \sum_{L \in \Omega} \frac{\partial M_{K,L}}{\partial Z_J} W_{I,L} &= -\frac{\sqrt{-1}}{2} \sum_{L=(a,b) \in \Omega} \{ \sigma_{(p,a),(r,s)} Y_{qb} + \sigma_{(q,b),(r,s)} Y_{pa} \\ &\quad + \sigma_{(q,a),(r,s)} Y_{pb} + \sigma_{(p,b),(r,s)} Y_{qa} \} W_{I,L} \\ &= -\frac{\sqrt{-1}}{2^{1+\delta(i,j)}} \left\{ \sum_{1 \leq a \leq g} \sigma_{(p,a),(r,s)} \delta(q, j) R_{ia} + \sum_{1 \leq b \leq g} \sigma_{(p,b),(r,s)} \delta(q, i) R_{jb} \right. \\ &\quad \left. + \sum_{1 \leq a \leq g} \sigma_{(q,a),(r,s)} \delta(p, i) R_{ja} + \sum_{1 \leq b \leq g} \sigma_{(q,b),(r,s)} \delta(p, j) R_{ib} \right\}. \end{aligned}$$

While

$$\begin{aligned} \sum_{1 \leq a \leq g} \sigma_{(p,a),(r,s)} \delta(q, j) R_{ia} &= \sum_{1 \leq a \leq g} \{ \delta(p, r) \delta(a, s) + \delta(p, s) \delta(a, r) \\ &\quad - \delta(p, r) \delta(a, s) \delta(p, s) \delta(a, r) \} \delta(q, j) R_{ia} \\ &= \delta(q, j) \{ \delta(p, r) R_{is} + \delta(p, s) R_{ir} - \delta(p, r) \delta(p, s) R_{is} \}, \\ \sum_{1 \leq b \leq g} \sigma_{(p,b),(r,s)} \delta(q, i) R_{jb} &= \delta(q, i) \{ \delta(p, r) R_{js} + \delta(p, s) R_{jr} - \delta(p, r) \delta(p, s) R_{js} \}, \\ \sum_{1 \leq a \leq g} \sigma_{(q,a),(r,s)} \delta(p, i) R_{ja} &= \delta(p, i) \{ \delta(q, r) R_{js} + \delta(q, s) R_{jr} - \delta(q, r) \delta(q, s) R_{js} \}, \\ \sum_{1 \leq b \leq g} \sigma_{(q,b),(r,s)} \delta(p, j) R_{ib} &= \delta(p, j) \{ \delta(q, r) R_{is} + \delta(q, s) R_{ir} - \delta(q, r) \delta(q, s) R_{is} \}. \end{aligned}$$

Combining these equalities together, we have

$$\begin{aligned} \sum_{L \in \Omega} \frac{\partial M_{K,L}}{\partial Z_J} W_{I,L} &= -\frac{\sqrt{-1}}{2^{1+\delta(i,j)}} \{ \delta(q, j) \{ \delta(p, r) R_{is} + \delta(p, s) R_{ir} - \delta(p, r) \delta(p, s) R_{is} \} \\ &\quad + \delta(q, i) \{ \delta(p, r) R_{js} + \delta(p, s) R_{jr} - \delta(p, r) \delta(p, s) R_{js} \} \\ &\quad + \delta(p, i) \{ \delta(q, r) R_{js} + \delta(q, s) R_{jr} - \delta(q, r) \delta(q, s) R_{js} \} \\ &\quad + \delta(p, j) \{ \delta(q, r) R_{is} + \delta(q, s) R_{ir} - \delta(q, r) \delta(q, s) R_{is} \} \} \end{aligned}$$

Similarly, we have

$$\begin{aligned} \sum_{L \in \Omega} \frac{\partial M_{K,L}}{\partial Z_I} W_{J,L} &= -\frac{\sqrt{-1}}{2^{1+\delta(r,s)}} \{ \delta(q,s) \{ \delta(p,i) R_{rj} + \delta(p,j) R_{ri} - \delta(p,i) \delta(p,j) R_{rj} \} \\ &\quad + \delta(q,r) \{ \delta(p,i) R_{sj} + \delta(p,j) R_{si} - \delta(p,i) \delta(p,j) R_{sj} \} \\ &\quad + \delta(p,r) \{ \delta(q,i) R_{sj} + \delta(q,j) R_{si} - \delta(q,i) \delta(q,j) R_{sj} \} \\ &\quad + \delta(p,s) \{ \delta(q,i) R_{rj} + \delta(q,j) R_{ir} - \delta(q,i) \delta(q,j) R_{rj} \} \} \end{aligned}$$

Finally we get

Lemma 3.2.

$$\begin{aligned} \Gamma_{I,J}^K &= -\frac{1}{2} \left(\sum_{L \in \Omega} \frac{\partial M_{K,L}}{\partial Z_J} W_{I,L} + \sum_{L \in \Omega} \frac{\partial M_{K,L}}{\partial Z_I} W_{J,L} \right) \\ &= \frac{\sqrt{-1}}{2^{2+\delta(i,j)}} \{ \delta(q,j) \{ \delta(p,r) R_{is} + \delta(p,s) R_{ir} - \delta(p,r) \delta(p,s) R_{is} \} \\ &\quad + \delta(q,i) \{ \delta(p,r) R_{js} + \delta(p,s) R_{jr} - \delta(p,r) \delta(p,s) R_{js} \} \\ &\quad + \delta(p,i) \{ \delta(q,r) R_{js} + \delta(q,s) R_{jr} - \delta(q,r) \delta(q,s) R_{js} \} \\ &\quad + \delta(p,j) \{ \delta(q,r) R_{is} + \delta(q,s) R_{ir} - \delta(q,r) \delta(q,s) R_{is} \} \} \\ &\quad + \frac{\sqrt{-1}}{2^{2+\delta(r,s)}} \{ \delta(q,s) \{ \delta(p,i) R_{rj} + \delta(p,j) R_{ri} - \delta(p,i) \delta(p,j) R_{rj} \} \\ &\quad + \delta(q,r) \{ \delta(p,i) R_{sj} + \delta(p,j) R_{si} - \delta(p,i) \delta(p,j) R_{sj} \} \\ &\quad + \delta(p,r) \{ \delta(q,i) R_{sj} + \delta(q,j) R_{si} - \delta(q,i) \delta(q,j) R_{sj} \} \\ &\quad + \delta(p,s) \{ \delta(q,i) R_{rj} + \delta(q,j) R_{ir} - \delta(q,i) \delta(q,j) R_{rj} \} \} \end{aligned}$$

Using the lemma 3.2 above, one can easily show that in the case $K = (p, q) = (1, 1)$, $I = (i, j) = (1, j)$, $J = (r, s) = (1, s)$, we have

- if $j = s = 1$, then $\Gamma_{I,J}^K = \sqrt{-1} R_{1,1}$;
- if $j = 1, s \neq 1$, then $\Gamma_{I,J}^K = \sqrt{-1} R_{1,s}$;
- if $j \neq 1, s \neq 1$, then $\Gamma_{I,J}^K = \sqrt{-1} R_{j,s} = \sqrt{-1} R_{s,j}$.

In general case, if $K = (p, q)$, $I = (i, j)$, $J = (r, s)$, and if both (Z_{ij}, Z_{rs}) and (Z_{rs}, Z_{ij}) do not belong to $\{Z_{1p}, Z_{2p}, \dots, Z_{gp}\} \times \{Z_{1q}, Z_{2q}, \dots, Z_{gq}\}$, then all terms in the last equality of lemma 3.2 are zero, hence $\Gamma_{I,J}^K = \Gamma_{J,I}^K = 0$.

If $p = q$, then

$$\begin{aligned} \Gamma_{I,J}^{(p,p)} &= \frac{\sqrt{-1}}{2^{1+\delta(i,j)}} \{ \delta(p,i) \{ \delta(p,r) R_{js} + \delta(p,s) R_{jr} - \delta(p,r) \delta(p,s) R_{jr} \} \\ &\quad + \delta(p,j) \{ \delta(p,r) R_{is} + \delta(p,s) R_{ir} - \delta(p,r) \delta(p,s) R_{ir} \} \} \\ &\quad + \frac{\sqrt{-1}}{2^{1+\delta(r,s)}} \{ \delta(p,r) \{ \delta(p,i) R_{sj} + \delta(p,j) R_{si} - \delta(p,i) \delta(p,j) R_{sj} \} \\ &\quad + \delta(p,s) \{ \delta(p,i) R_{rj} + \delta(p,j) R_{ri} - \delta(p,i) \delta(p,j) R_{rj} \} \}. \end{aligned}$$

If $Z_{ij} = Z_{ji}$ and $Z_{rs} = Z_{sr}$ belong to the same row or column with Z_{pp} , then $i = r = p$, or $i = s = p$, or $j = r = p$, or $j = s = p$.

- If $i = r = p$, one can use the formula above to show that $\Gamma_{I,J}^K = \sqrt{-1} R_{js}$
- If $i = s = p$, then $\Gamma_{I,J}^K = \sqrt{-1} R_{jr}$.
- If $j = r = p$, then $\Gamma_{I,J}^K = \sqrt{-1} R_{is}$.
- If $j = s = p$, then $\Gamma_{I,J}^K = \sqrt{-1} R_{ir}$.

If $p < q$, we may assume that $Z_{ij}(i \leq j)$ belong to the same row with Z_{pq} and $Z_{rs}(r \leq s)$ belongs to the same column with Z_{pq} (Other cases can be proved in the same way). Then $i = p \leq j$, $r \leq s = q$ and

$$\begin{aligned}\Gamma_{I,J}^K &= \frac{\sqrt{-1}}{2^{2+\delta(i,j)}}(\delta(q,j)\delta(p,r)R_{is} + R_{jr} + \delta(p,j)R_{ir}) \\ &\quad + \frac{\sqrt{-1}}{2^{2+\delta(r,s)}}(\delta(p,r)\delta(q,j)R_{si} + R_{rj} + \delta(q,r)R_{sj}).\end{aligned}$$

- If $i = p = j \leq q$ and $r < s = q$, then

$$\Gamma_{I,J}^K = \frac{\sqrt{-1}}{8}(R_{jr} + R_{ir}) + \frac{\sqrt{-1}}{4}R_{rj} = \frac{\sqrt{-1}}{2}R_{jr}.$$

- If $i = p = j$ and $r = s = q$, then $\Gamma_{I,J}^K = \frac{\sqrt{-1}}{2}R_{jr}$.
- If $i = p < j$ and $r < s = q$, then

$$\begin{aligned}\Gamma_{I,J}^K &= \frac{\sqrt{-1}}{4}(\delta(q,j)\delta(p,r)R_{is} + R_{jr}) + \frac{\sqrt{-1}}{4}(\delta(p,r)\delta(q,j)R_{is} + R_{rj}) \\ &= \frac{\sqrt{-1}}{2}(\delta(q,j)\delta(p,r)R_{pq} + R_{jr}) \\ &= \begin{cases} \frac{\sqrt{-1}}{2}R_{pq} & \text{if } j = q \text{ and } r = p, \\ \frac{\sqrt{-1}}{2}R_{jr} & \text{otherwise.} \end{cases}\end{aligned}$$

- If $i = p < j$ and $r = s = q$, then $\Gamma_{I,J}^K = \frac{\sqrt{-1}}{2}R_{jr}$.

At last, we complete the proof of Lemma 2.1.

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